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Algebraic solutions of an sl -boson system in the $U(2l + 1) \longleftrightarrow O(2l + 2)$ transitional region

Feng Pan^{1,2}, Xin Zhang¹ and J P Draayer²

¹ Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China

² Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

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Abstract

Exact eigen-energies and the corresponding wavefunctions of an interacting sl -boson system in a $U(2l + 1) \longleftrightarrow O(2l + 2)$ transitional region are obtained by using the Bethe ansatz within an infinite-dimensional Lie algebra. A numerical algorithm for solving the Bethe ansatz equations is introduced. As an example, spectra for the $U(3) \longleftrightarrow O(4)$ transitional region of the $U(4)$ Vibron model are analysed.

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1. Introduction

Many physical systems can be described, at least approximately, by interacting bosons. Examples can be found in interacting Bose–Einstein condensates [1, 2], low-energy nuclear models [3–5] and theories of molecular structure [6–8]. For example, in the interacting sd -boson model for nuclei valence nucleon pairs are treated approximately as s and d bosons. In this case the Hamiltonian of the system is constructed in terms of $U(6)$ generators. Generally, as an extension, the largest dynamical symmetry group generated by s and l ($l = p, d, f, \dots$) boson operators is $U(2l + 2)$ when the total number of bosons is a conserved quantity. If only one- and two-body interactions are introduced, and the angular momentum of the sl -boson system is conserved, the Hamiltonian of the model can be expressed in terms of a linear combination of the first- and second-order Casimir operators of subalgebras contained in all possible chains of the reduction $U(2l + 2) \downarrow O(3)$,

$$\hat{\mathbf{H}} = A_1 \hat{\mathbf{C}}_1(U(2l + 2)) + A_2 \hat{\mathbf{C}}_2(U(2l + 1)) + A_3 \hat{\mathbf{C}}_2(O(2l + 2)) + \dots + B \hat{\mathbf{C}}_2(O(2l + 1)) + C \hat{\mathbf{C}}_2(O(3)). \quad (1.1)$$

In this paper, we only consider a special case, namely, when the Hamiltonian is constructed out of a linear combination of the first- and second-order Casimir operators of $U(2l + 2)$, $O(2l + 2)$, $U(2l + 1)$, $O(2l + 1)$ and $O(3)$. Other possible chains in the reduction

$U(2l+2) \downarrow O(3)$ will not be considered. Specifically, those terms represented by the ellipsis in (1.1), which represent Casimir operators up to second order of other subgroups of $U(2l+2)$, will not be taken into account. In this case the Hamiltonian can be diagonalized in one of the following chains:

$$\begin{array}{ccc}
 & \nearrow U(2l+1) \searrow & \\
 U(2l+2) & & O(2l+1) \supset O(3). \\
 & \searrow O(2l+2) \nearrow &
 \end{array} \quad (1.2)$$

Obviously, (1.1) is diagonal under the $U(2l+2) \supset U(2l+1) \supset O(2l+1) \supset O(3)$ chain when $A_3 = 0$, while it is also diagonal under another chain $U(2l+2) \supset O(2l+2) \supset O(2l+1) \supset O(3)$ when $A_2 = 0$. These two special cases are called the $U(2l+1)$ and $O(2l+2)$ limits, respectively. If all the parameters are nonzero, the system is in the $U(2l+1) \longleftrightarrow O(2l+2)$ transitional region. Obviously, exact diagonalization of (1.1) in the transitional region is not as easy as in either of the limits, especially when the dimension of the configuration space is relatively large. However, the Hamiltonian (1.1) can still be diagonalized numerically by using generators of the $SU(1, 1)$ Lie algebra outlined in [9, 10].

Recently, it has been pointed out that exact solutions of the interacting sd -boson model in the $U(5) \longleftrightarrow O(6)$ transitional region can also be obtained by using an algebraic Bethe ansatz within an infinite-dimensional Lie algebraic approach [11]. A similar procedure discussing s and d boson dominance in the interacting boson system was considered in [12]. Though numerical matrix diagonalization of the Hamiltonian (1.1) is not difficult, the algebraic Bethe ansatz procedure should be useful for similar quantum many-body problems, especially for some problems where straightforward diagonalization becomes infeasible.

In this paper, the method outlined in [11] will be extended and applied to sl -boson systems, which provides an example to illustrate the algebraic Bethe ansatz method. It should be pointed out that the $l = 1$ case of the model is the $U(4)$ vibron model proposed by Iachello *et al* in describing rotation–vibration modes of diatomic and triatomic molecules [6–8] and α -cluster excitations in nuclei [13]. In contrast with earlier applications, the method proposed in this paper can be used to discuss transitional situations.

In section 2, an algebraic Bethe ansatz method for diagonalizing the Hamiltonian (1.1) is introduced. Solutions of the latter are determined within the framework of an infinite-dimensional Lie algebra. In section 3, a numerical algorithm for solving the Bethe ansatz equations is presented. And in section 4, as an example, spectra of an sp -boson system in the transitional region are considered.

2. $\widehat{SU}(1, 1)$ algebra and exact solutions

The $SU^l(1, 1)$ generators can be constructed in terms of l -boson operators as

$$S^+(l) = \frac{1}{2}l^\dagger \cdot l^\dagger \quad S^-(l) = \frac{1}{2}\tilde{l} \cdot \tilde{l} \quad S^0(l) = \frac{1}{2} \left(l^\dagger \cdot \tilde{l} + \frac{2l+1}{2} \right). \quad (2.1)$$

Similarly, s -boson operators can be used to construct the $SU^s(1, 1)$ algebra, the generators of which can be written as

$$S^+(s) = \frac{1}{2}s^{\dagger 2} \quad S^-(s) = \frac{1}{2}s^2 \quad S^0(s) = \frac{1}{2} \left(s^\dagger s + \frac{1}{2} \right). \quad (2.2)$$

Furthermore, generators of the $SU^{sl}(1, 1)$ algebra can be constructed from both s - and l -boson operators with

$$S^\mu(sl) = S^\mu(l) \pm S^\mu(s) \quad (2.3)$$

where $\mu = 0, +, -$, and either the $+$ or the $-$ sign can be taken.

As is well known the basis vectors of $U(2l+1) \supset O(2l+1)$ and $O(2l+2) \supset O(2l+1)$ are also those of $SU^l(1, 1) \supset U^l(1)$ and $SU^{sl}(1, 1) \supset U^{sl}(1)$, respectively [10, 11]. Their duality relations are shown as follows:

$$|Nn_l v n_\Delta LM\rangle = \left| N, \kappa^l = \frac{1}{2} \left(v + \frac{2l+1}{2} \right), \mu^l = \frac{1}{2} \left(n_l + \frac{2l+1}{2} \right), n_\Delta LM \right\rangle \quad (2.4)$$

for basis vectors of $U(2l+1) \supset O(2l+1)$ and those of $SU^l(1, 1) \supset U^l(1)$, where N, n_l, v, L and M are quantum numbers of $U(2l+2), U(2l+1), O(2l+1), O(3)$ and $O(2)$, respectively, n_Δ is an additional quantum number needed in the reduction $O(2l+1) \downarrow O(3), \kappa^l$ and μ^l are quantum numbers of $SU^l(1, 1)$ and $U^l(1)$, respectively. Therefore, basis vectors of $U(2l+2) \supset O(2l+2) \supset O(2l+1) \supset O(3)$ are also those of $SU^{sl}(1, 1)$ and $SU^l(1, 1)$, simultaneously, which can be expressed as

$$|N\sigma v n_\Delta LM\rangle = \left| N, \kappa^{sl} = \frac{1}{2}(\sigma + l + 1), \mu^{sl} = \frac{1}{2}(N + l + 1), \kappa^l = \frac{1}{2} \left(v + \frac{2l+1}{2} \right), n_\Delta LM \right\rangle \quad (2.5)$$

where σ is the quantum number of $O(2l+2)$, κ^{sl} and μ^{sl} are the quantum numbers of $SU^{sl}(1, 1)$ and its subalgebra $U^{sl}(1)$, respectively.

Next we introduce operators

$$S_n^\pm = c_s^{2n+1} S^\pm(s) + c_l^{2n+1} S^\pm(l) \quad S_n^0 = c_s^{2n} S^0(s) + c_l^{2n} S^0(l) \quad (2.6)$$

where c_s and c_l are real, and n can be taken to be $0, \pm 1, \pm 2, \dots$. It can be verified that these operators satisfy the following commutation relations:

$$[S_m^0, S_n^\pm] = \pm S_{m+n}^\pm \quad [S_m^+, S_n^-] = -2S_{m+n}^0. \quad (2.7)$$

According to the definitions $\{S_m^\mu, \mu = 0, \pm 1, \pm 2, \dots; m = 0, \pm 1, \pm 2, \dots\}$ generate the affine Lie algebra $\widehat{SU}(1, 1)$ without central extension.

Let $|lw\rangle$ be the lowest weight state of $\widehat{SU}(1, 1)$, which should satisfy

$$S^-(s)|lw\rangle = 0 \quad S^-(l)|lw\rangle = 0. \quad (2.8)$$

Since the total number of bosons, N , is a conserved quantity, and the basis vectors are restricted under the subalgebra chain $O(2l+1) \supset O(3) \supset O(2)$, the lowest weight states satisfying (2.8) are actually a set of basis vectors of $U(2l+2) \supset U(2l+1) \supset O(2l+1) \supset O(3) \supset O(2)$ with

$$|lw\rangle = \left| N; \kappa^l = \frac{1}{2} \left(v + \frac{2l+1}{2} \right), \mu^l = \frac{1}{2} \left(n_l + \frac{2l+1}{2} \right), \kappa^s = \frac{1}{2} \left(v_s + \frac{1}{2} \right), \right. \\ \left. \mu^s = \frac{1}{2} \left(n_s + \frac{1}{2} \right); n_\Delta LM \right\rangle \quad (2.9)$$

where $N = v + v_s, n_l = v, n_s = v_s = 0$ or 1 . Hence, we have

$$S_n^0|lw\rangle = \Lambda_n^0|lw\rangle \quad (2.10)$$

where

$$\Lambda_n^0 = \frac{1}{2} \left[c_s^{2n} \left(v_s + \frac{1}{2} \right) + c_l^{2n} \left(v + \frac{2l+1}{2} \right) \right]. \quad (2.11)$$

Using the generators of $\widehat{SU}(1, 1)$, we can construct a Hamiltonian for the $U(2l+1) \longleftrightarrow O(2l+2)$ transitional region as

$$\widehat{H} = g S_0^+ S_0^- + \alpha S_1^0 + \gamma \widehat{C}_2(O(2l+1)) + \delta \widehat{C}_2(O(3)) \quad (2.12)$$

where g, α, γ and δ are real parameters. It can be seen that (2.12) is equivalent to a Hamiltonian in the $O(2l+2)$ limiting case when $c_s = c_l$, and to a Hamiltonian in the $U(2l+1)$ limiting case when $c_s = 0$ and $c_l \neq 0$. Hence, the general $c_s \neq c_l \neq 0$ cases correspond to the $U(2l+1) \longleftrightarrow O(2l+2)$ transitional situations. In the following, c_l is fixed, and we allow c_s to vary within the closed interval $[0, c_l]$. It should be pointed out that the first two terms in the Hamiltonian (2.12) will become a special case of the generalized pairing Hamiltonian with two orbits discussed in [14] when the $SU(1, 1)$ generators $S^\mu(l)$ and $S^\mu(s)$ are replaced by two sets of $SU(2)$ generators for two different orbits, which is the extension of Richardson's pairing model discussed early in [15–18].

To diagonalize the Hamiltonian (2.12), we use the algebraic Bethe ansatz which implies that eigenvectors of (2.12) may be expressed as

$$|k; \nu_s \nu_n \Delta LM\rangle = \mathcal{N} S^+(x_1) S^+(x_2) \cdots S^+(x_k) |l w\rangle. \quad (2.13)$$

In this expression $S^+(x_i)$ ($i = 1, 2, \dots, k$) is a functional operator of $S^+(l)$ and $S^+(s)$ with spectral parameter x_i , since the subspace with fixed total number of bosons is spanned by $\{(S^+(l))^{k-\mu} (S^+(s))^\mu |l w\rangle\}$ ($\mu = 0, 1, 2, \dots, k$), and \mathcal{N} is a normalization factor. However, the explicit form of $S^+(x_i)$ and allowed values of x_i in (2.13) need to be determined. To determine the explicit functional form and allowed values of x_i , we first expand (2.13) in terms of the spectral parameters x_i around $x_i \sim 0$,

$$|k; \nu_s \nu_n \Delta LM\rangle = \mathcal{N} \sum_{n_i \in \mathbb{Z}} a_{n_1} a_{n_2} \cdots a_{n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} S_{n_1}^+ S_{n_2}^+ \cdots S_{n_k}^+ |l w\rangle \quad (2.14)$$

where

$$a_{n_i} S_{n_i}^+ = \frac{1}{2\pi i} \oint_0 dx_i x_i^{n_i} S^+(x_i) \quad (2.15)$$

is the Fourier–Laurent coefficient in the expansion of $S^+(x_i)$. Directly solving the eigenvalue problem of (2.12) with expansion (2.14), one gets two parts, of which the one proportional to S_0^+ is not of the original ansatz form given by (2.14), and should be set to zero, and the other part can be written as

$$\begin{aligned} & (\hat{h} - \alpha \Lambda_1^0) \sum_{n_i} a_{n_1} a_{n_2} \cdots a_{n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} S_{n_1}^+ \cdots S_{n_k}^+ |l w\rangle \\ &= \alpha \sum_{\mu} \sum_{n_i} a_{n_1} \cdots a_{n_{\mu-1}} a_{n_{\mu-1}} a_{n_{\mu+1}} \cdots a_{n_k} \frac{1}{x_\mu} S_{n_1}^+ \cdots S_{n_k}^+ |l w\rangle \end{aligned} \quad (2.16)$$

where $\hat{h} \equiv \hat{H} - \gamma \hat{C}_2(O(2l+1)) - \delta \hat{C}_2(O(3))$. Hence, one gets

$$h^{(k)} - \alpha \Lambda_1^0 = \sum_{\mu=1}^k \frac{\alpha a_{n_{\mu-1}}}{x_\mu a_{n_\mu}}. \quad (2.17)$$

Since $h^{(k)}$ should be an eigenvalue of \hat{h} , which should be n_μ ($\mu = 1, 2, \dots, k$) independent, the only possible situation is that

$$\frac{a_{n_{\mu-1}}}{a_{n_\mu}} = \kappa_\mu \quad (2.18)$$

for $\mu = 1, 2, \dots, k$, where κ_μ is a μ -dependent constant. Conditions (2.18) are equivalent to rescaling the spectral parameters x_i by

$$y_i = x_i / \kappa_i. \quad (2.19)$$

As a result of (2.19), one has

$$a_{n_i} x_i^{n_i} = y_i^{n_i} \quad (2.20)$$

for $i = 1, 2, \dots, k$, where a_0 has been set to 1. Therefore, one can equivalently choose $a_{n_i} = 1$. Hence, the explicit form of the functional $S^+(x_i)$ is

$$S^+(x_i) = \frac{c_s}{1 - c_s^2 x_i} S^+(s) + \frac{c_l}{1 - c_l^2 x_i} S^+(l). \quad (2.21)$$

A similar form to (2.21) was first used by Gaudin as an ansatz in finding exact solutions of a spin–spin interaction system [19], which is now proved to be a consistent operator form in constructing the Bethe ansatz wavefunction (2.13) for the current interacting boson system.

Using the commutation relations

$$\begin{aligned} [S_1^0, S^+(x)] &= \frac{1}{x} S^+(x) - \frac{1}{x} S_0^+ \\ [[S_0^-, S^+(x)], S^+(y)] &= \frac{2}{x-y} (S^+(x) - S^+(y)) \\ [S_0^-, S^+(x)] &= \frac{c_s^2 S^0(s)}{1 - c_s^2 x} + \frac{c_l^2 S^0(l)}{1 - c_l^2 x_i} \end{aligned} \quad (2.22)$$

one can derive the eigenequation with

$$\begin{aligned} &(\hat{h} - \alpha \Lambda_1^0) S^+(x_1) \cdots S^+(x_k) |lw\rangle \\ &= \alpha \sum_i S^+(x_1) \cdots S^+(x_{i-1}) \left(\frac{1}{x_i} S^+(x_i) - \frac{1}{x_i} S_0^+ \right) S^+(x_{i+1}) \cdots S^+(x_k) |lw\rangle \\ &\quad - g S_0^+ \sum_i S^+(x_1) \cdots S^+(x_{i-1}) S^+(x_{i+1}) \cdots S^+(x_k) \sum_{j \neq i} \frac{2}{x_i - x_j} |lw\rangle \\ &\quad + g S_0^+ \sum_i \left(\frac{c_s^2 (v_s + \frac{1}{2})}{1 - c_s^2 x_i} + \frac{c_l^2 (v + \frac{2l+1}{2})}{1 - c_l^2 x_i} \right) \\ &\quad \times S^+(x_1) \cdots S^+(x_{i-1}) S^+(x_{i+1}) \cdots S^+(x_k) |lw\rangle. \end{aligned} \quad (2.23)$$

In (2.23), terms proportional to S_0^+ should be set to zero, which leads to a set of Bethe ansatz equations that determines the spectral parameters x_i :

$$\frac{\alpha}{x_i} = \frac{g c_s^2 (v_s + \frac{1}{2})}{1 - c_s^2 x_i} + \frac{g c_l^2 (v + \frac{2l+1}{2})}{1 - c_l^2 x_i} - \sum_{j \neq i} \frac{2g}{x_i - x_j} \quad i = 1, 2, \dots, k. \quad (2.24)$$

Therefore, the infinite-dimensional algebraic expansion (2.14) along with the energy eigenequation completely determines the functional $S^+(x_i)$ and the possible values that the spectral parameters x_i can take on. It should be noted that even though the functional $S^+(x_i)$ is expanded around $x_i \sim 0$, the results should also be valid on the entire complex plane because of the analytical behaviour of the eigenvectors. Therefore, results (2.21) and (2.24) are solutions of the problem. The eigenvalues $E^{(k)}$ of the Hamiltonian (2.12) can be expressed as

$$E^{(k)} = h^{(k)} + \gamma v(v + 2l - 1) + \delta L(L + 1) + \alpha \Lambda_1^0 \quad (2.25)$$

where

$$h^{(k)} = \sum_{i=1}^k \frac{\alpha}{x_i} \quad (2.26)$$

and

$$\Lambda_1^0 = \frac{1}{2} \left[c_s^2 \left(v_s + \frac{1}{2} \right) + c_l^2 \left(v + \frac{2l+1}{2} \right) \right]. \quad (2.27)$$

The quantum number k is related to the total number of bosons N by

$$N = 2k + \nu_s + \nu. \quad (2.28)$$

Since the total number of bosons is conserved, the quantum number ν_s is not independent and can be omitted. Generally, there are many different solutions of equation (2.24). If there are p sets of solutions, an additional quantum number, $\zeta = 1, 2, \dots, p$, should be introduced to distinguish them from one another. Thus, wavefunction (2.13) is denoted as $|k; \zeta; \nu n_{\Delta} LM\rangle$. It should be noted that for a k -pair excitation the spectral parameters x_i have S_k symmetry. Any permutations among different roots x_i for $i = 1, 2, \dots, k$, are also solutions of (2.24), which can be seen from (2.13) and (2.26). Therefore, in general, there will be $k!$ different solutions with the same eigen-energy and the corresponding wavefunction, the only difference being that the spectral parameters, x_i , are interchanged. These solutions should be regarded as repeated roots with only one of them being a solution to the problem.

3. Numerical algorithm

Though eigen-energies and the corresponding wavefunctions can easily be determined for limiting cases, in general cases equation (2.24) is a set of non-linear Bethe ansatz equations (BAE) with k unknowns for a k -pair excitation. Obtaining the BAE for an exactly solvable system is only half of the solution. The difficulty is that quantities of physical interest, eigen-energies, etc, are expressed in terms of solutions of the BAE; and unfortunately, the BAE are generally hard to solve. Many methods have been developed for extracting solutions from these equations. A typical example for solving the BAE in the Hubbard model with nearest neighbour hopping was shown in [20]. Exact solutions of a similar set of equations to (2.24) were also discussed in [21] by Shastry and Dhar, which is a generalized version of the Stieltjes problem [22]. In comparison to the generalized Stieltjes equation considered in [21], there are more complicated terms, namely, the first two terms on the right-hand side of equation (2.24) instead of a constant in the generalized Stieltjes equation. Therefore, the method used in [21] is not directly applicable for finding exact solutions of (2.24). On the other hand, an analysis similar to that used in [21] is helpful in understanding the behaviour of the solutions. In this section, a useful and simple numerical algorithm for solving the BAE (2.24) using MATHEMATICA will be outlined. Also, because of the S_k symmetry with respect to permutations among the $\{x_1, x_2, \dots, x_k\}$, in the following we exclude those solutions that can be obtained by such root permutations, keeping only one since the others correspond to the same eigen-energy and wavefunction.

Equation (2.24) can be rewritten as

$$\frac{\beta}{y_i} = \frac{c^2 (v_s + \frac{1}{2})}{1 - c^2 y_i} + \frac{(v + \frac{2l+1}{2})}{1 - y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j} \quad i = 1, 2, \dots, k \quad (3.1)$$

where

$$\beta = \alpha/g \quad c = c_s/c_l \leq 1 \quad y_i = c_l^2 x_i. \quad (3.2)$$

Firstly, we need to know how many different sets of solutions of equation (3.1) exist, excluding those that can be obtained by root permutations of the S_k . Assume that the total number of such solutions is p . The roots in (3.1) can then be arranged as $\{y_1^{(\zeta)}, y_2^{(\zeta)}, \dots, y_k^{(\zeta)}\}$ with $\zeta = 1, 2, \dots, p$. Each set of roots corresponds to unique eigenvectors. These eigenvectors span a subspace, which is called the diagonalized configuration subspace. The s -boson Hamiltonian is diagonal in the diagonalized configuration subspace. Let us use

$s^q l^r$ ($q+r=k$) symbolically to denote a k -pair excitation configuration with q s -boson pairs and r l -boson pairs. Then these $k+1$ k -pair excitations $\{s^k l^0, s^{k-1} l^1, \dots, s^0 l^k\}$ span a non-diagonalized configuration subspace, namely, the sl -boson Hamiltonian, in general, is not diagonal in the non-diagonalized configuration subspace. The basis vectors of the diagonalized configuration subspace can be obtained from those of the non-diagonalized one by a unitary transformation

$$S^q L^r = \sum_{q'r'} a_{q'r'}^{qr} s^{q'} l^{r'} \quad (3.3)$$

where $S^q L^r$ symbolically denotes the basis vectors of diagonalized configuration subspace, $a_{q'r'}^{qr}$ is the transformation coefficient. The transformation must be an one-to-one full mapping which cannot change dimension of the space. Thus, we come to a conclusion that the dimension of the diagonalized configuration subspace is also $k+1$, of which the basis vectors are denoted symbolically as $\{S^k L^0, S^{k-1} L^1, \dots, S^0 L^k\}$. Furthermore, it can easily be seen that, for fixed i , equation (3.1) will become a polynomial equation of order $k+1$, which should have $k+1$ different roots with $x_i \neq x_j$ for $j \neq i$, and there are $k!$ permutations among these k different roots which will lead to $k!$ solutions. This fact justifies that solutions of the Bethe ansatz (3.1) are complete and equivalent to numerical diagonalization in the non-diagonalized configuration space. Therefore, we have the following theorem:

Theorem. For k -pair excitation, besides solutions that can be obtained by root permutations of the S_k , equation (3.1) has $k+1$ different sets of solutions, namely $p = k+1$.

According to this theorem, for a k -pair excitation equation (3.1) has $(k+1)!$ solutions, while only $k+1$ of them are solutions to the eigenvalue problem of the sl -boson system.

In what follows, we use a concrete example to show how to numerically find all roots of equation (3.1). We take the sd -boson system with a k -pair excitation as an example, in which the parameters $v_s = 1$, $v_d = 2$, $\beta = 1000$ and $c = 0.2$. Thus, equation (3.1) becomes

$$\frac{1000}{y_i} = \frac{0.06}{1 - 0.04y_i} + \frac{4.5}{1 - y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j} \quad i = 1, 2, \dots, k. \quad (3.4)$$

We use notation $[k]$ to denote equation (3.4). It can be verified that solutions of $[k+1]$ can be obtained by using FindRoot in MATHEMATICA from those for $[k]$. Starting from $[k=1]$, which can easily be calculated by using Solve in MATHEMATICA, one can get solutions of $[k=2]$. Hence, an iterative procedure can be set up for finding solutions of $[m+1]$ from those of $[m]$. Excluding roots that can be obtained from S_k symmetry, all inequivalent roots up to $[k=4]$ are listed in table 1. It can easily be seen from table 1 that the roots of $[k]$ have the following properties: (1) the two roots of $[k=1]$ are very different; (2) two roots of $[k=1]$ are fundamental because all other roots of $[k]$ are in the vicinity of these two roots; (3) assume that the bigger root of $[k=1]$ corresponds to the basis vector S and smaller one corresponds to the basis vector D in the two-dimensional diagonalized configuration subspace. Then, there are q relatively larger roots and r smaller roots of $[k]$ corresponding to the basis vector $S^q D^r$ ($q+r=k$) in the k -dimensional diagonalized subspace. These properties can be used to set up a procedure for finding all inequivalent roots of $[k]$ using MATHEMATICA.

For example, as shown in table 1, the roots of $[k=2]$ and $[k=1]$ are related as follows:

$$\begin{cases} y_1^{(1)}(k=2) \gtrsim y^{(1)}(k=1) & \begin{cases} y_1^{(2)}(k=2) \sim y^{(1)}(k=1) \\ y_2^{(2)}(k=2) \sim y^{(2)}(k=1) \end{cases} & \begin{cases} y_1^{(3)}(k=2) \gtrsim y^{(2)}(k=1) \\ y_2^{(3)}(k=2) \lesssim y^{(2)}(k=1) \end{cases} \end{cases} \quad (3.5)$$

Table 1. Solutions of the Bethe ansatz equation (3.4).

k	N	$y_i^{(\zeta)}$ ($i = 1, 2, \dots, k; \zeta = 1, 2, \dots, k + 1$)	$S^q D^r$ ($q + r = k$)
1	5	$y^{(1)} = 24.9627$ $y^{(2)} = 0.99552$	$S^1 D^0$ $S^0 D^1$
2	7	$\begin{cases} y_1^{(1)} = 24.9772 \\ y_2^{(1)} = 24.8988 \end{cases}$	$S^2 D^0$
		$\begin{cases} y_1^{(2)} = 24.9628 \\ y_2^{(2)} = 0.99552 \end{cases}$	$S^1 D^1$
		$\begin{cases} y_1^{(3)} = 0.99686 \\ y_2^{(3)} = 0.99221 \end{cases}$	$S^0 D^2$
3	9	$\begin{cases} y_1^{(1)} = 24.9835 \\ y_2^{(1)} = 24.9306 \\ y_3^{(1)} = 24.8261 \end{cases}$	$S^3 D^0$
		$\begin{cases} y_1^{(2)} = 24.9772 \\ y_2^{(2)} = 24.899 \\ y_3^{(2)} = 0.995519 \end{cases}$	$S^2 D^1$
		$\begin{cases} y_1^{(3)} = 24.9629 \\ y_2^{(3)} = 0.99686 \\ y_3^{(3)} = 0.99221 \end{cases}$	$S^1 D^2$
		$\begin{cases} y_1^{(4)} = 0.997548 \\ y_2^{(4)} = 0.994224 \\ y_3^{(4)} = 0.988891 \end{cases}$	$S^0 D^3$
		$\begin{cases} y_1^{(1)} = 24.987, & y_2^{(1)} = 24.9466 \\ y_3^{(1)} = 24.873, & y_4^{(1)} = 24.7488 \end{cases}$	$S^4 D^0$
		$\begin{cases} y_1^{(2)} = 24.9835, & y_2^{(2)} = 24.9307 \\ y_3^{(2)} = 24.8264, & y_4^{(2)} = 0.995519 \end{cases}$	$S^3 D^1$
4	11	$\begin{cases} y_1^{(3)} = 24.9773, & y_2^{(3)} = 24.8992 \\ y_3^{(3)} = 0.99686, & y_4^{(3)} = 0.992209 \end{cases}$	$S^2 D^2$
		$\begin{cases} y_1^{(4)} = 24.963, & y_2^{(4)} = 0.997548 \\ y_3^{(4)} = 0.994224, & y_4^{(4)} = 0.98889 \end{cases}$	$S^1 D^3$
		$\begin{cases} y_1^{(5)} = 0.997979, & y_2^{(5)} = 0.995339 \\ y_3^{(5)} = 0.99145, & y_4^{(5)} = 0.985541 \end{cases}$	$S^0 D^4$

while the relationships of the roots of $[k = 3]$ with those of $[k = 2]$ are

$$\begin{cases} y_1^{(1)}(k=3) \gtrsim y_1^{(1)}(k=2) \\ y_2^{(1)}(k=2) < y_2^{(1)}(k=3) < y_1^{(1)}(k=2) \\ y_3^{(1)}(k=3) \lesssim y_2^{(1)}(k=2) \end{cases} \quad \begin{cases} y_1^{(2)}(k=3) \sim y_1^{(1)}(k=2) \\ y_2^{(2)}(k=3) \sim y_2^{(1)}(k=2) \\ y_3^{(2)}(k=3) \sim y_2^{(2)}(k=2) \end{cases} \quad (3.6)$$

$$\begin{cases} y_1^{(3)}(k=3) \sim y_1^{(2)}(k=2) \\ y_2^{(3)}(k=3) \sim y_1^{(3)}(k=2) \\ y_3^{(3)}(k=3) \sim y_2^{(3)}(k=2) \end{cases} \quad \begin{cases} y_1^{(4)}(k=3) \gtrsim y_1^{(3)}(k=2) \\ y_2^{(3)}(k=2) < y_2^{(4)}(k=3) < y_1^{(3)}(k=2) \\ y_3^{(4)}(k=3) \lesssim y_2^{(3)}(k=2) \end{cases} .$$

From this type of analysis one sees that the roots of $[k = m + 1]$ are in the vicinity of those of $[k = m]$. Generally, the solutions of $[k = m + 1]$ can be obtained from different root

combinations of $[k = m]$, which can be calculated according to the combinations of S and D components in the diagonalized configuration subspace. Although the above analysis is based on a special numerical example, the conclusions apply to the cases with $\beta > 0$, and $0 < c < 1$ as well. In all of these cases the roots of equation (3.1) are real.

Using these properties, we can set up a procedure to find all roots of equation (2.24) numerically by using MATHEMATICA. Firstly, one can use Solve to get solutions of $[k = 1]$. Then one can use the combination of roots corresponding to the basis vector S and the corresponding basis vector D to get all roots of $[k = 2]$. It is obvious that the roots of $[k]$ are all different, $y_1 \neq y_2 \neq \dots \neq y_k$. In this procedure, we always assume that the relatively smaller roots correspond to the S component, and the larger ones correspond to the D component in the diagonalized configuration subspace, which can be used as a guideline for finding the solution. In order to find solutions for $[k = 3]$ from those of $[k = 2]$, for example, a solution $\{y_1, y_2, y_3\}$ of $[k = 3]$ corresponding to the basis vector $S^2 D^1$ can be found by using $\{x_1, x_2\}$ of $[k = 2]$ corresponding to S^2 and x of $[k = 1]$ corresponding to D as initial roots. One can then use FindRoot with these initial roots to get a new solution. Similarly, solutions for $[k]$ corresponding to the basis vector $S^r D^q$ with $r + q = k$ can be obtained either from the roots of $[k - 1]$ corresponding to the basis vector $S^{r-1} D^q$ and the root of $[k = 1]$ corresponding to the basis vector S , or from the roots of $[k - 1]$ corresponding to the basis vector $S^r D^{q-1}$ and the root of $[k = 1]$ corresponding to the basis vector D , as initial roots.

4. Complete solutions of the $U(4)$ vibron model

In this section, we will use the results outlined in the previous two sections to discuss the sp -boson system, of which limiting cases were discussed thoroughly in [6–8]. There are only two different reductions in this case, $U(4) \supset O(4) \supset O(3) \supset O(2)$ and $U(4) \supset U(3) \supset O(3) \supset O(2)$, with basis vectors denoted as $|N\sigma\nu M\rangle$ and $|Nn_p\nu M\rangle$, respectively. Because $l = 1$, the quantum number $L = \nu$ in this case.

In the molecular vibron model the two limiting cases, $U(4) \supset O(4) \supset O(3) \supset O(2)$ and $U(4) \supset U(3) \supset O(3) \supset O(2)$, correspond to rigid and non-rigid situations, respectively. However, the rigid and non-rigid cases are two idealized limits. In the real world no molecule is absolutely rigid or non-rigid. If the $O(4)$ and $U(3)$ limiting cases are regarded as the rigid phase and non-rigid phase, respectively, the $O(4) \longleftrightarrow U(3)$ transitional region is where the two phases coexist. Within the framework of the vibron model, the vibrational–rotational modes of diatomic molecules are described by a $O(4) \longleftrightarrow U(3)$ transitional Hamiltonian. The parameter c in equation (3.1) is called the phase parameter since the $c = 1$ case corresponds to the $O(4)$ limit, while the $c = 0$ case corresponds to the $U(3)$ limit. Firstly, we can calculate the energy spectrum of the model with the fixed phase parameter c . In table 2, the Bethe ansatz solutions of an example with $N = 6$, $c = 0.2$, $\alpha = 1000$, $g = 1$, $c_p = 1$ are given.

Because $L = \nu$, the two terms $\nu(\nu + 2l - 1)$ and $L(L + 1)$ in (2.25) can be merged into one with a new parameter $\eta = \nu + \delta$. Using solutions of Bethe ansatz, one can get the eigen-energies and the corresponding wavefunctions. Tables 3 and 4 show eigen-energies with $\eta = 0$ and $\eta = 100$, respectively, while other parameters are the same as those used in table 2. In these two tables the notation ν_ζ is used to label energy levels, while ν is the angular momentum quantum number, and ζ is used to distinguish from different levels with the same other quantum numbers and different solutions of equation (3.1).

In the following we fix the parameter $\eta = 100$, and allow the phase parameter c to vary within the closed interval $[0, 1]$. Other parameters are the same as those used in table 2.

Table 2. Solutions of (3.1) with $N = 6$, $c = 0.2$, $\alpha = 1000$, $g = 1$, $c_p = 1$.

ν	k	$y_i^{(\zeta)}$ ($i = 1, 2, \dots, k; \zeta = 1, 2, \dots, k+1$)	Configuration
			$S^q P^r$ ($q+r = k$)
0	3	(1)24.8628, 24.9556, 24.9953	$S^3 P^0$
		(2)24.9323, 24.9932, 0.998 502	$S^2 P^1$
		(3)24.9876, 0.995 934, 0.999 083,	$S^1 P^2$
		(4)0.993 011, 0.997 211, 0.999 336	$S^0 P^3$
1	2	(1)24.8986, 24.9771	$S^2 P^0$
		(2)24.9627, 0.997 506	$S^1 P^1$
		(3)0.994 655, 0.998 376	$S^1 P^2$
2	2	(1)24.9323, 24.9932	$S^2 P^0$
		(2)24.9876, 0.996 512	$S^1 P^1$
		(3)0.993 418, 0.997 631	$S^1 P^2$
3	1	(1)24.9627	$S^1 P^0$
		(2)0.995 52	$S^0 P^1$
4	1	(1)24.9876	$S^1 P^0$
		(2)0.994 53	$S^0 P^1$

Table 3. Eigen-energies E (in arbitrary units) with $c = 0.2$ and $\eta = 0$.

ν_ζ	0_1	0_2	0_3	0_4	1_1	1_2	1_3
E	0	961.321	1924.72	2890.2	479.901	1442.26	2406.7
ν_ζ	2_1	2_2	2_3	3_1	3_2	4_1	4_2
E	959.821	1923.22	2888.7	1439.76	2404.2	1919.72	2885.2

Table 4. Eigen-energies E (in arbitrary units) with $c = 0.2$ and $\eta = 100$.

ν_ζ	0_1	0_2	0_3	0_4	1_1	1_2	1_3
E	0	961.321	1924.72	2890.2	679.901	1642.26	2606.7
ν_ζ	2_1	2_2	2_3	3_1	3_2	4_1	4_2
E	1559.821	2523.22	3488.7	2639.76	3604.2	3919.72	4885.2

Numerical results for eigen-energies are shown in table 5. The transitional spectrum from one phase to the other for the phase parameter c varying within the closed interval $[0, 1]$ is shown in figure 1. It can be seen from figure 1 that level degeneracy occurs when c takes some special values, especially when $c = 0.7$ – 0.8 . The degeneracy seems accidental. However, the system must have special symmetry at these special points corresponding to the accidental degeneracy. Possible connection to the $E(2l+1)$ symmetry [23–25], which is related to a special symmetry in the $U(2l+1) \leftrightarrow SO(2l+2)$ transitional region discussed in this paper, needs to be further explored.

One can also investigate the influence on the eigen-energies of other parameters. For example, in order to show how the eigen-energies vary with the parameter β within the whole range of the phase parameter c , the energy surface $E(c, \beta)$ with the other parameters fixed can be defined and calculated using our procedure. In figure 2 the energy surfaces, $E_{1_1^-}(c, \beta)$, $E_{2_1^+}(c, \beta)$, $E_{3_1^-}(c, \beta)$, $E_{4_1^+}(c, \beta)$ are shown, where other fixed parameters are $N = 6$, $g = 1$, $c_p = 1$ and $\eta = 100$. From these figures, it is clear that the transition from $U(3)$ to $O(4)$ is second order phase transitional.

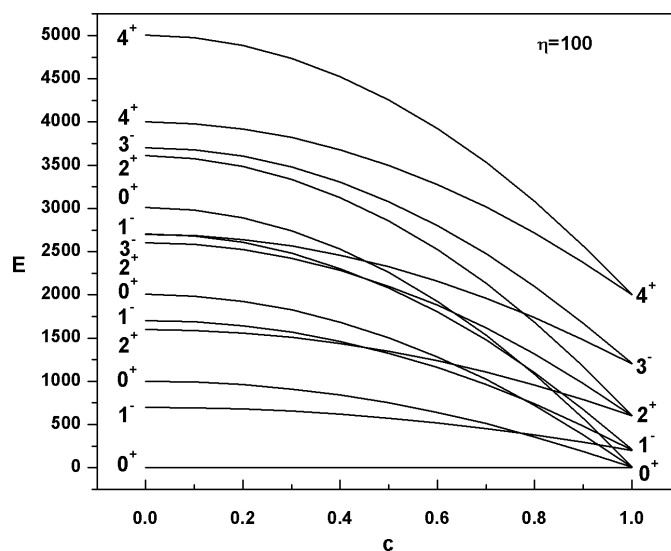


Figure 1. Some low-lying energy levels of the vibron model in the whole transitional region with $N = 6$, $\beta = 1000$, $\eta = 100$, where phase parameter $c = 0$ corresponds to the $U(3)$ limit, and $c = 1$ to the $O(4)$ limit.

Table 5. Low-lying energy levels (in arbitrary units) with different phase parameter c in the vibron model.

c	$E(0_1)$	$E(0_2)$	$E(0_3)$	$E(0_4)$	$E(1_1)$	$E(1_2)$	$E(1_3)$
0.0	0	1001.5	2005	3010.5	700	1702.5	2707
0.1	0	991.455	1984.94	2980.43	694.975	1687.45	2681.93
0.2	0	961.32	1924.72	2890.2	679.9	1642.26	2606.7
0.3	0	911.1	1824.38	2739.83	654.78	1566.97	2481.33
0.4	0	840.78	1683.88	2529.3	619.6	1461.54	2305.8
0.5	0	750.38	1503.26	2258.63	574.38	1326.01	2080.13
0.6	0	639.89	1282.5	1927.81	519.1	1160.35	1804.32
0.7	0	509.31	1021.59	1536.84	453.78	964.58	1478.35
0.8	0	358.63	720.56	1085.73	378.4	738.68	1102.24
0.9	0	187.89	379.42	574.49	292.97	482.69	676.01
1.0	0	6	10	12	200	206	210
c	$E(2_1)$	$E(2_2)$	$E(2_3)$	$E(3_1)$	$E(3_2)$	$E(4_1)$	$E(4_2)$
0.0	1600	2603.5	3609	2700	3704.5	4000	5005.5
0.1	1589.96	2583.44	3578.93	2684.94	3679.43	3979.93	4975.43
0.2	1559.82	2523.22	3488.7	2639.76	3604.2	3919.72	4885.2
0.3	1509.6	2422.88	3338.33	2564.46	3478.83	3819.37	4734.83
0.4	1439.28	2282.38	3127.8	2459.04	3303.3	3678.88	4524.3
0.5	1348.88	2101.76	2857.13	2323.5	3077.63	3498.25	4253.63
0.6	1238.38	1880.99	2526.31	2157.85	2801.81	3277.49	3922.81
0.7	1107.8	1620.09	2135.34	1962.07	2475.85	3016.58	3531.84
0.8	957.12	1319.05	1684.23	1736.17	2099.73	2715.54	3080.73
0.9	786.36	977.9	1172.99	1480.16	1673.5	2374.36	2569.48
1.0	600	606	610	1200	1206	2000	2006

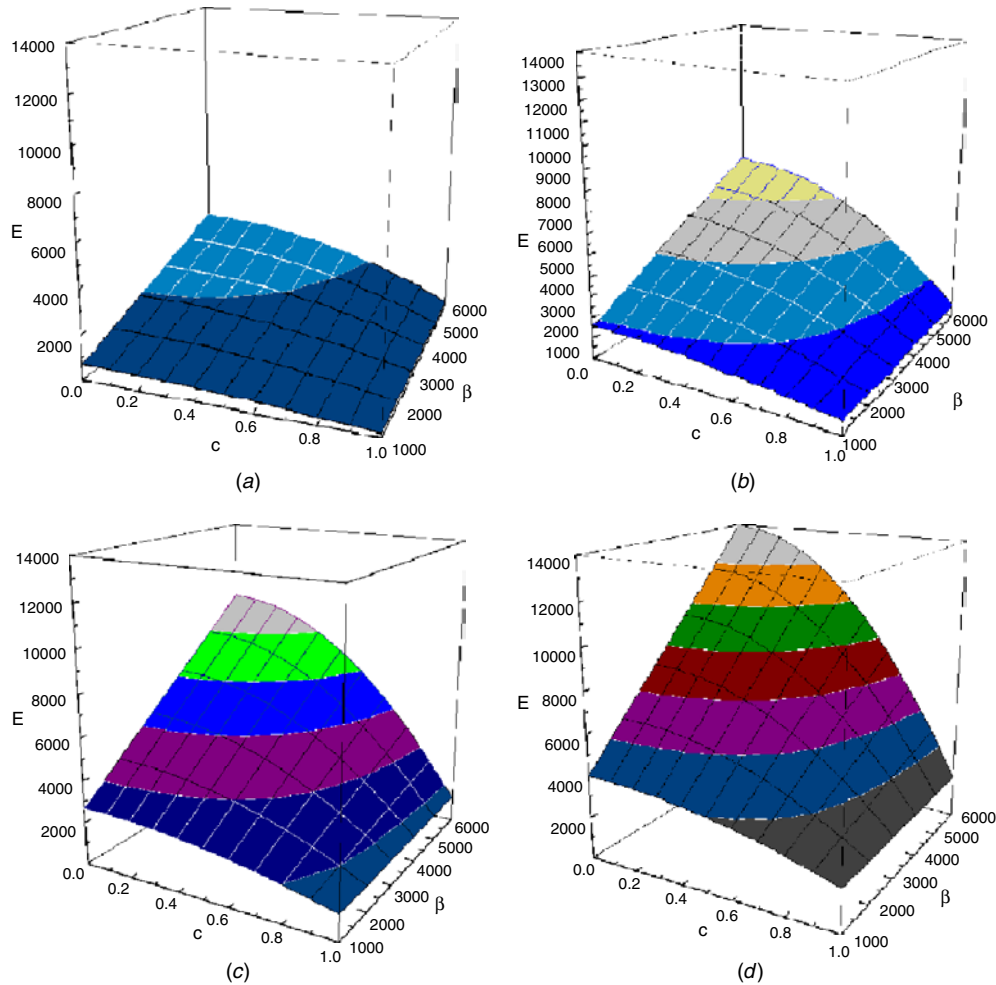


Figure 2. Some low-lying energy surfaces (in arbitrary units) in the two-phase coexistence region. (a) Energy surface of 1_1^- , (b) energy surface of 2_1^+ , (c) energy surface of 3_1^- and (d) energy surface of 4_1^+ .

5. Conclusion

In this paper, the interacting sl -boson system in the $U(2l+1) \longleftrightarrow O(2l+2)$ transitional region is exactly solved. The total numbers of bosons and angular momentum were considered to be conserved quantities and only one- and two-body interactions were included in the Hamiltonian. To find exact solutions of the problem, an algebraic Bethe ansatz was introduced. In order to determine the functionals $S^+(x_i)$ and allowed values of the spectral parameters x_i , an infinite-dimensional Lie algebraic method was introduced, which can be used to determine the functionals $S^+(x_i)$ as well as the Bethe ansatz equations, of which the solutions provide the allowed values of these spectral parameters. Physical quantities, such as eigen-energies and their corresponding wavefunctions can be evaluated in terms of these spectral parameters.

In order to make the procedure practical for realistic applications, a numerical algorithm for solving the Bethe ansatz equations was introduced. Specifically, FindRoot of the

MATHEMATICA package can be invoked to find the roots of the non-linear equation. As an example, some energy spectra for the $U(3) \longleftrightarrow O(4)$ transitional region of the $U(4)$ vibron model were studied. As the results demonstrated, the procedure should be useful in studying transitional situations in the vibrational–rotational modes in molecules and α -cluster excitations in nuclei. Although the eigenvalue problem studied in this paper can also be solved easily by the direct diagonalization procedure, the method outlined in this paper should be useful for other quantum many-body problems, in which numerical diagonalization will become infeasible.

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